From Dispersion to Laminarity in Dynamical Systems *

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We study transport in dynamical systems characterized by intermittent chaotic behavior with coexistence of dispersive motion due to periods of localization, and of enhanced diffusion due to periods of laminar motion. This transport is discussed within the continuous-time random walk approach which applies to both dispersive and enhanced motions. We analyze the coexistence for the standard map and for a one-dimensional map.

I. Introduction

One of the fascinating problems in the theory of statistical physics has been the description of stochastic processes generated by deterministic dynamical systems and the understanding of the complicated motion in such systems [1–8]. In particular it has been observed that particles' orbits do not necessarily obey simple Gaussian behavior, rather they may exhibit an intermittent laminar chaotic behavior which results in dispersive or enhanced diffusional motion [3–10].

Examples of anomalous diffusion in dynamical systems cover both dissipative and Hamiltonian systems. One finds diffusion anomalies in numerical studies of one-dimensional maps [4, 5, 9], of the Chirikov-Taylor standard map [1, 2, 10, 11], of stochastic webs [7], and of conservative motion in two-dimensional potential surfaces [6, 12]. Experimentally, anomalies have been observed for tracer diffusion in flow systems [13–15].

The statistical description of anomalous diffusion can be formulated in terms of continuous-time random walks for dispersive motion and for enhanced diffusion [4, 9]. Broad distributions of waiting times have been shown to lead to dispersive transport [4, 9]. Lévy stable laws have been proposed as a method for introducing enhancement. These distributions generalize the central limit theorem and therefore naturally result in the generalization of Brownian motion [8, 9, 16].

Originally, Lévy flights were introduced in the search for distributions for which the shape of the spatial probability distribution, the propagator, is independent of the number of steps performed in time t. This requirement is naturally fulfilled by Gaussian distributions. The generalization of the Gaussian case leads to the Lévy stable distributions, $L_\gamma(x)$; for $\gamma = 2$ we recover the Gaussian distribution. For $\gamma < 2$ the large scale behavior is characterized by a power law $L_\gamma(x) \sim |x|^{-\gamma-1}$, and consequently the variance diverges, or more generally $\langle |x|^2 \rangle = \infty$ for $\gamma < 2$. As mentioned the displacement grows faster than that in Brownian motion, namely $\langle |x| \rangle \sim t^{1/\gamma}$ for $1 < \gamma \le 2$. This property has made the Lévy flights natural candidates for the description of enhanced diffusion [17–19].

In diffusion problems the mean-squared displacement $\langle |x|^2 \rangle$ is usually considered as the central quantity in the calculation of transport properties. Here a difficulty arises due to the fact that for the Lévy stable laws the variance of a single step diverges, and thus also the mean-squared displacement diverges at any time instance. Thus the description of enhanced diffusion in terms of stable laws is controversial. Ways to avoid this difficulty are to introduce cut-offs in the distribution [20] or to concentrate on typical displacements rather than on the moments. Another method, which has recently been proposed, is based on a more realistic picture; it accounts for the fact that the velocity of the particle under consideration is finite. Thus the particle's motion is restricted by its velocity. Assuming for instance, apart from turning points, a constant velocity v, one has $|x| \le v t$. This observation has stimulated the introduction of a time cost into the single event probability distribution [9, 16]

$$\psi(x, t) = \delta(|x| - v t)\, \psi(t), \tag{1}$$

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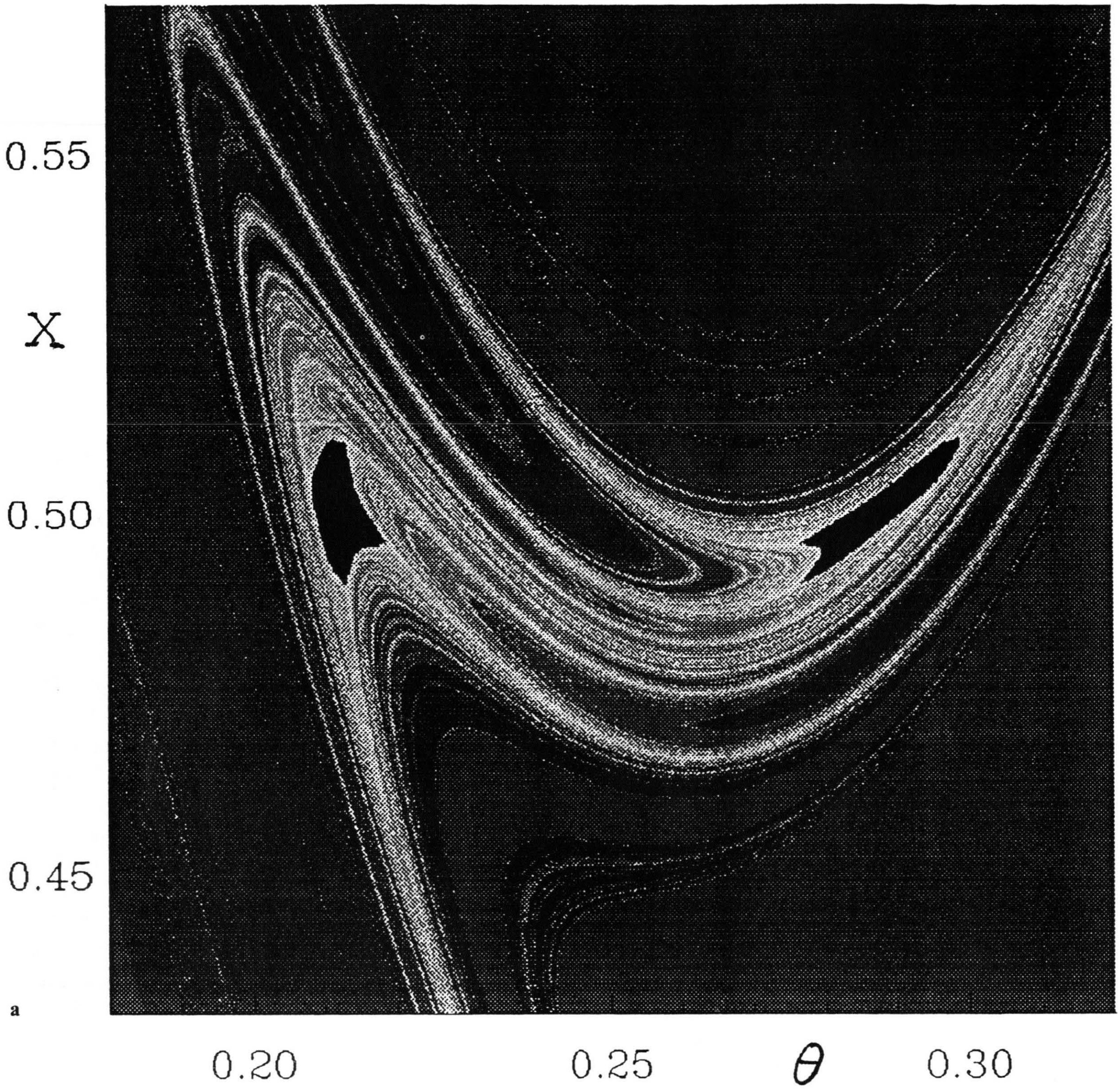
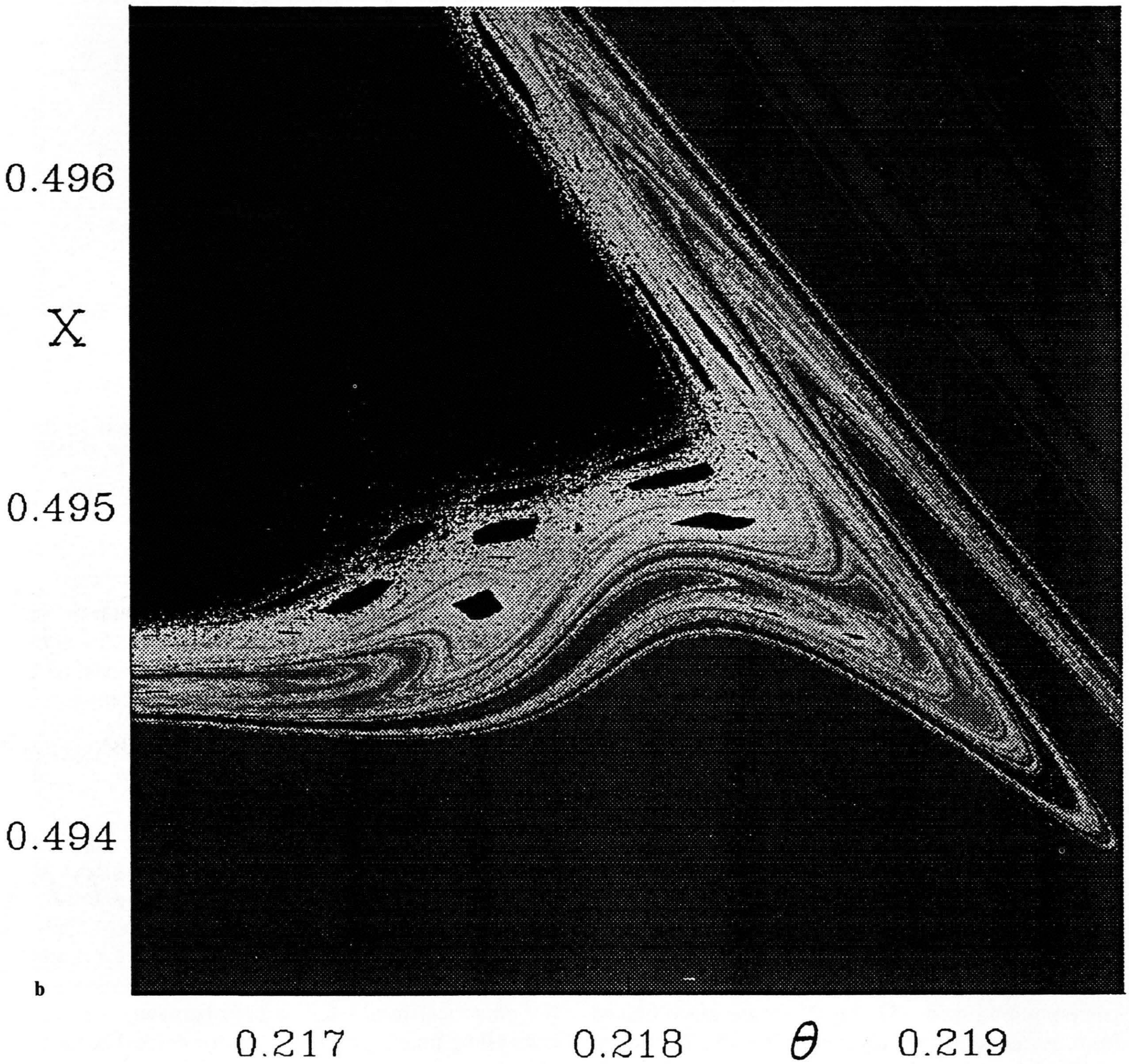


Fig. 1. Escape times for the standard map for the localizing $p = 2$, $l = 0$ mode for $K = 1.03$. The two black areas centered around $x = 0.5$ are islands of stability. The logarithms of the escape times are plotted on gray scales: bright shading denote long escape times, dark areas are due to chaotic motion. Figure 1b shows a window centered onto the right corner of the left island. The nested structure of the islands is obvious.

where $\psi(t)$ denotes the probability densities for being locked in a laminar mode for time t. $\psi(x,t)$ is the probability density to move in a single motion event a distance x in time t; the δ-function accounts for the constant velocity. The single event probability distribution given in (1) has been shown to be useful in the description of enhanced diffusion in dissipative systems such as iterated non-linear maps [9] and of motion in conservative dynamical systems [11, 12].

A different approach originates from the generalized thermodynamic formalism and is based on generalized partition functions [21, 22]. In what follows we consider continuous-time random walks with power-law waiting time distributions which lead to anoma-



b

lous diffusion, and we exemplify possible coexistences of the two types of anomalies.

II. Coexistence of modes of localization and of laminarity

Trajectories in dynamical systems show in many cases a complicated intermittent behavior: the motion can be governed by periods of no motion (localization), of chaotic, or of laminar motion. For instance such a behavior has been observed for fluids in a rotating annulus [15]. Tracer particles stick temporarily to eddies and after release they are temporarily locked in a mode of laminar motion. The times of sticking to the eddies and of being in a laminar phase have been found to be approximately distributed as power-laws [15].

A useful example which shows a similar behavior is provided by the standard map [1, 2, 10, 11]. Here the motion is dictated by the iteration scheme

$$x_{n+1} = x_n + K \sin(2\pi\theta_n),$$
$$\theta_{n+1} = \theta_n + x_{n+1} \pmod 1, \qquad (2)$$

where θ is an angle and K is the stochasticity parameter [2]. Depending on the stochasticity parameter for values above a critical value $K_c = 0.971\ldots$ the trajectories may show a pronounced intermittency which gives rise to enhanced diffusional behavior. The superlinear increase of $\langle x^2(t)\rangle$ originates from temporal resonances with the accelerating modes. The corresponding resonance times follow broad distributions and the broadness relates to the hierarchical structure of the islands of stability [17].

The stability islands arise around fixed points which can be classified according to two arguments p and l: p denotes the period, so that $\theta_{n+p} = \theta_n$, and l denotes the displacement, so that $|x_{n+p} - x_n| = l$. Hierarchies of stability islands can show up in the vicinity of basic $p - l$ fixed points [10]. The fundamental accelerating modes $p = l$ are found to be particularly efficient in the enhancement of the diffusion. Enhanced diffusion due to temporal resonances with the $p = l = 3$ ($K \simeq 1.1$) and with the $p = l = 5$ fundamental modes have been observed [10, 11].

It is not surprising that different modes of stability may coexist for particular values of K. For instance for $K = 1.03$ the fundamental mode $p = l = 5$ centered around $x \simeq 0$ [10, 11] coexists with the $p = 2$, $l = 0$ mode which is centered around $x \simeq 0.5$. In Figs. 1a and 1b the island system of the localizing $p = 2$, $l = 0$ mode is shown. Plotted are the escape times, the times required for a trajectory initiated in the neighborhood of the stability islands to get out of resonance with the corresponding mode [11, 23]. The data are obtained from the numerical realization of (2) with a maximum of 10^6 iterations. This number of iterations does not provide a sufficient resolution to fully unravel the nested system of islands, from the Figure, however, it is obvious that islands exist on many scales. Another method of visualization is to plot the trajectories initiated in the area of interest; this method provides only qualitative information while escape times can be considered for a statistical study [11, 23].

We have analyzed the distribution $\tilde{\psi}(t)$ of sticking times for the $p = 2$, $l = 0$ mode from a few long trajectories of typically 10^{11} iterations. The results are shown in Fig. 2 and are compared with $\psi(t)$, the probability distribution of being locked in the fundamental

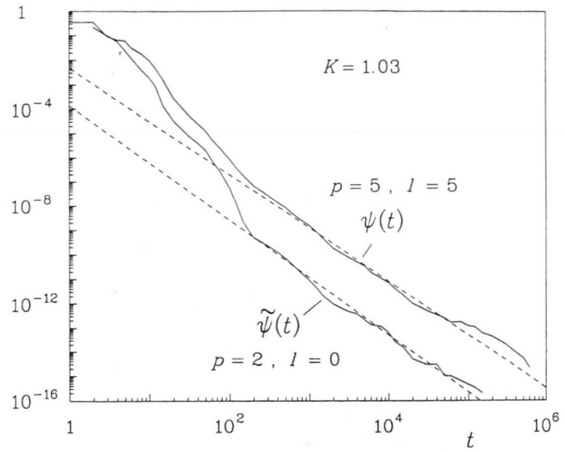


Fig. 2. Temporal resonances with modes of stability for the standard map. Plotted are the probability densities of trapping times by full lines for the $p = 2$, $l = 0$ and $p = l = 5$ modes, as indicated. The dashed lines give the slopes according to the exponents $\gamma = 1.2$ and $\tilde{\gamma} = 1.3$.

accelerating mode $p = l = 5$ [11]. In our analysis we have assumed power laws for the two probability densities: $\tilde{\psi}(t) \sim t^{-\tilde{\gamma}-1}$ and $\psi(t) \sim t^{-\gamma-1}$. Interestingly, the fitted exponents are very similar to each other, namely $\tilde{\gamma} = 1.3$ and $\gamma = 1.2$. Considering the escape times instead of the long trajectories in the analysis, we have found the exponents being even closer to each other.

The two broad distributions indicate strongly that the motion is intermittently governed by modes of localization and by modes of laminar motion. Results about the propagator have been studied previously, here we present the mean-squared displacement and the findings are displayed in Figure 3. We notice that the numerical results follow approximately a power law at long times, $\langle x^2(t)\rangle \sim t^\alpha$ with $\alpha = 1.8$. The value of the exponent α is compatible with the relationship $\alpha = 3 - \gamma$, which will be discussed below.

A similar situation has been encountered for the conservative motion in the two-dimensional potential $V(x, y) = A + B(\cos x + \cos y) + C \cos x \cos y$ [6, 12]. Depending on the energy E of the trajectory there are pronounced periods of laminar, coaxial motion due to a channeling effect. However, these periods are often interrupted by long periods of localization. In fact a hierarchy of islands of stability for localizing modes has been detected in the corresponding Poincaré maps [24]. One may therefore conclude that the interplay between modes of localization and of laminar motion

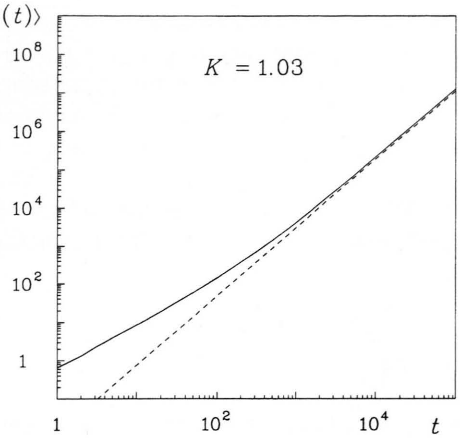


Fig. 3. Time evolution of the mean-squared displacement for the standard map. The numerical results are given for $K = 1.03$ by the full line. The dashed line is the slope according to the adjusted exponent $\alpha = 1.8$.

is a common phenomenon in dynamical systems. Obviously, the restriction to only two different types of modes in the description is an approximation.

In order to demonstrate the combined behavior in a model which can be analytically studied we have introduced a one-dimensional map which shows both, temporal localization and laminar motion [25]. The iteration follows the scheme

$$f(x) = \begin{cases} (1+\varepsilon)\,x + a\,x^{z} - 1\,, & 0 \le x \le \frac{1}{4}, \\ (1+\tilde{\varepsilon})\,x - \tilde{a}\,(\frac{1}{2} - x)^{\tilde{z}}, & \frac{1}{4} \le x \le \frac{1}{2}. \end{cases} \tag{3}$$

Translational and inversion symmetries are inferred so that diffusional motion from one to the other unit cell is enabled. This map represents a combination of two maps introduced previously [4, 5, 9]; an example is presented in Figure 4. The map is discontinuous at the boundaries of each box but is continuous otherwise. For $\varepsilon = \tilde{\varepsilon} = 0$ the map shows marginal stable fixed points at $x = 0$ and $x = 1/2$. The neighborhood of the fixed point at $x = 0$ is responsible for the laminarity, while the fixed point at $x = 1/2$ gives rise to localization. The exponents z and $\tilde{z}$ determine the characteristic behaviors of the two types of events, free flights and localization, respectively. The prefactors a and $\tilde{a}$ are weights and are chosen so that the map $f(x)$ is continuous at $x = 1/4$ including the first derivative. For $\varepsilon = \tilde{\varepsilon} = 0$ we used $a = 4^{z}\,\tilde{z}/(z + \tilde{z})$ and $\tilde{a} = 4^{\tilde{z}}\,z/(z + \tilde{z})$. ε and $\tilde{\varepsilon}$ were considered to be small quantities and to differ from zero in the cases $z > 2$ and $\tilde{z} > 2$, respectively, in order to avoid problems in the numerical realization of the statistical analyses.

The relatively simple map of (3) displays, depending on the two exponents z and $\tilde{z}$, a rich spectrum of behaviors and a unique interplay of the two modes of motion. It thus mimics the behavior mentioned above, namely the interplay of being localized in a unit cell and of being locked in a mode of laminar motion. Assuming that the events of localization and of laminar motion are uncorrelated one can analyze the iterated map in terms of the continuous-time random walk approach [25]. As its main ingredients we reconsider the distributions $\tilde{\psi}(t)$, the probability density of being stuck in a unit cell for t iterations, and the distribution $\psi(t)$, the probability of moving from one to the next unit cell in a laminar mode for t iterations. Both distributions follow asymptotically power laws: $\tilde{\psi}(t) \sim t^{-\tilde{\gamma}-1}$ and $\psi(t) \sim t^{-\gamma-1}$. The characteristic exponents are related to those of the map through: $\tilde{\gamma} = 1/(\tilde{z} - 1)$ and $\gamma = 1/(z - 1)$.

We previously investigated propagators and mean-squared displacements in various dynamical systems; here we concentrate on the time evolution of the displacement. Because the behavior depends on two exponents we primarily distinguish between two cases: $\tilde{\gamma} < 1$ and $\tilde{\gamma} > 1$. In the case $\tilde{\gamma} > 1$ the average sticking time to a unit cell is finite; as a consequence the periods of localization do not alter the behavior dictated

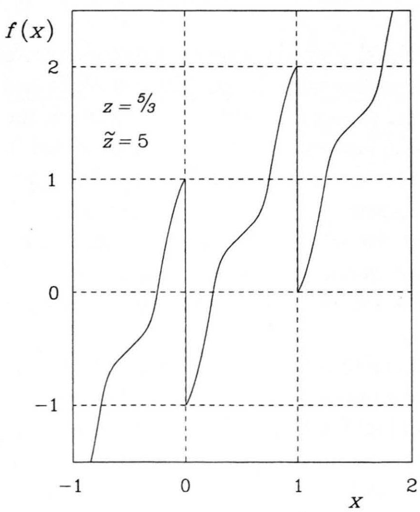


Fig. 4. The map function $f(x)$ for the laminar-localized motion, Eq. (3), for $z = 5/3$ and $\tilde{z} = 5$, as indicated, corresponding to the exponents $\gamma = 3/2$ and $\tilde{\gamma} = 1/4$.

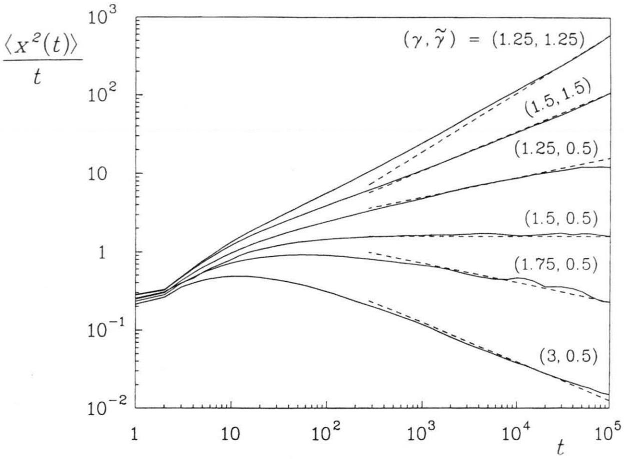


Fig. 5. Mean-squared displacement for the one dimensional iterated map of Eq. (3). Plotted are the numerical results of $\langle x^2(t)\rangle/t$ as full lines, the dashed lines give the predicted behavior according to Eqs. (4) and (5).

by the periods of laminar motion. One thus has [5, 9, 16, 25]

$$\langle x^2(t)\rangle \sim \begin{cases} t^2, & \gamma < 1 \\ t^{3-\gamma}, & 1 < \gamma < 2, \quad \tilde{\gamma} > 1, \\ t, & \gamma > 2 \end{cases} \tag{4}$$

In the case $0 < \tilde{\gamma} < 1$ the mean time of being localized within a unit-cell diverges; therefore the mean-squared displacement shows a dispersive aspect. Our analysis gives [25]

$$\langle x^2(t)\rangle \sim t^{2+\tilde{\gamma}-\gamma}, \quad 1 < \gamma < 2, \quad 0 < \tilde{\gamma} < 1. \tag{5}$$

We have examined $\langle x^2(t)\rangle$ numerically and have compared its behavior with the predictions of (4) and (5). The results are presented in Figure 5. Plotted is the ratio $\langle x^2(t)\rangle/t$ for various pairs of the exponents, γ and $\tilde{\gamma}$; the ratio $\langle x^2(t)\rangle/t$ has been chosen to highlight the dispersive aspect or the enhancement over the regular diffusion for which $\langle x^2(t)\rangle \sim t$. Both, dispersive behavior and enhancement are observed, and the long-time slopes are in agreement with the predictions.

Referring to the situation encountered above for the standard map, we notice that $\tilde{\gamma} > 1$, so that (4) applies and $\alpha = 3 - \gamma$. Thus for this map the asymptotic behavior of the mean-squared displacement is not influenced by the localizing mode. Contrary for the fluid motion in a rotating annulus: in [9] the exponents $\mu = 2.3 \pm 0.2$ for the flights, $\nu = 1.6 \pm 0.3$ for the sticking and $\alpha = 1.65 \pm 0.15$ for the mean-squared displacement are reported, which were determined from independent measurements. According to the present convention we have $\gamma = \mu - 1 = 1.3 \pm 0.2$ and $\tilde{\gamma} = \nu - 1 = 0.6 \pm 0.3$ so that (5) applies and therefore $\alpha = 1.3 \pm 0.5$, which is consistent with the experimental value.

III. Conclusions

In this paper we have demonstrated that dynamical systems may show competing effects in dominating the diffusional motion. We have presented examples for which the motion is governed simultaneously by slowing-down and by accelerating mechanisms. The motion in dynamical systems is complicated and its description in terms of alternating periods of no motion or laminar motion is an oversimplification of the reality. Nevertheless, this approach has shown to describe reasonably well the predominant behavior of the motion in these systems.

The dynamics generated by iterated maps or by Hamiltonian systems is deterministic. Because of the chaotic nature of the motion in these systems a stochastic approach is feasible which established the conditions for an analysis in terms of random walks. Previous methods of analysis have been extended to account for motion continuous in space and continuous in time; the method provides adequate means for the description of the observed intermittent chaotic motion. In this paper we have demonstrated its applicability to the dynamics of the standard map and of a one dimensional map. A complementary picture may be envisaged using also the generalized thermodynamic formalism, which is a promising approach for achieving information about the motion in dynamical systems.

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